For most practical purposes it is sufficient to determine γ to this order of accuracy in order to calculate the attenuation in the guide. If necessary, however, we can substitute this result back into (24) in order to determine $\psi^{(1)}$ and so calculate the fields in the guide to $O(\epsilon_1)$. These fields then supply boundary conditions via the continuity equations for the calculation of the next approximation to the fields in the conductor, which in turn can be used as boundary conditions for the determination of the next approximation to the fields in the guide, and so on. In carrying out this procedure the proper ordering of terms will be automatic with the field equations in the form given here.

V. Summary

A consistent approximation scheme for lossy waveguides has been developed which can, in a straightforward manner, be extended to any order of accuracy desired. Although two small parameters are involved in the expansion of the fields in the guide and its walls, the field equations are shown to force the correct ordering of parameters in this expansion.

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Expansions for the Capacitance of a Cross Concentric with a Circle with an Application

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***Abstract*—Expansions are given for the capacitance per unit length for the geometry having a cross section in which an equiarmed cross is concentric with an external circle.**

I. Introduction

Oberhettinger and Magnus [1, p. 61] have considered the problem of determining the capacitance of a coaxial structure whose outer conductor has a circular cross section while the cross section of the inner conductor is a line through the axis. The problem considered here differs in that the inner conductor has a cross section which is a symmetric cross centered on the axis of the outer circular conductor as shown in the z plane of Fig. 1.

The transformation

$$t=-\frac{1}{2}\left(z+\frac{1}{z}\right) \tag{1}$$

maps the upper left-hand quadrant of the circle in the z plane onto the upper right-hand quadrant of the t plane of Fig. 1. Here corresponding points on the boundaries of the two regions are given the same alphabetical name. Then the upper right-hand quadrant of the t plane is mapped onto the upper half of the w plane by the transformation

$$w=t^2 \tag{2}$$

Here again, the same letter is used to denote corresponding points. The capacitance C_0 of the coaxial structure is four times the capacitance, in the upper half w plane, between the line segment fa and the infinite line segment, bg. This capacitance, C, is given by the well-known formula [2, p. 58]

$$C=\frac{K(k)}{K'(k)} \tag{3}$$

where, in our case,

$$k^2=\frac{(a-f)(g-b)}{(g-a)(b-f)}. \tag{4}$$

Finally,

$$C_0=4\frac{K(k)}{K'(k)}. \tag{5}$$

In this paper, two series for C_0, one in terms of δ and the other in terms of $\rho=1-\delta$, are given which have certain theoretical and practical advantages. Not only do the series give the limiting behavior of C_0 as δ approaches 0 or 1 but they permit the direct calculation of C_0 with sufficient accuracy for most engineering applications without any resort to elliptic functions.

II. Analysis

If the values for a, b, f, and g, given in the w plane of Fig. 1, are substituted in (4), it is found that

$$k^2=\frac{8\delta^2(1+\delta^4)}{(1+\delta^2)^4}. \tag{6}$$

The values of C_0 given in the middle row of Table I were obtained by finding k^2 from (6), for the given values of δ, and then calculating the complete elliptic integrals, K and K' of (5), using Landen's transformation. To obtain an expansion for C_0 in terms of δ, one may expand (6) in terms of δ^2 to get

$$k^2=8\delta^2(1-4\delta^2+11\delta^4-24\delta^6+45\delta^8-76\delta^{10}+\cdots) \tag{7}$$

and then substitute in

$$\pi\frac{K'}{K}=\ln\frac{16}{k^2}-\frac{1}{2}\left(k^2+\frac{13}{32}k^4+\frac{23}{96}k^6+\frac{2701}{16384}k^8+\cdots\right) \tag{8}$$

so that finally

$$\pi\frac{K'}{K}=\ln 2-2\ln\delta-\frac{1}{8}\left(\delta^8+\frac{13}{32}\delta^{16}+\frac{23}{96}\delta^{24}+\frac{2701}{16384}\delta^{32}+\cdots\right). \tag{9}$$

Determining the coefficent of δ^{32} in (9) in this way requires the coefficents in (8) up to the coefficent of k^{32}. Those not given in [3, p. 1219] are provided in the Appendix.

In order to find the expansion for C_0 in powers of ρ, find k'^2 from (6) and replace δ with $1-\rho$. Then

$$k'^2=\rho^4\frac{1-\rho/2}{1-\rho+\rho^2/2} \tag{10}$$

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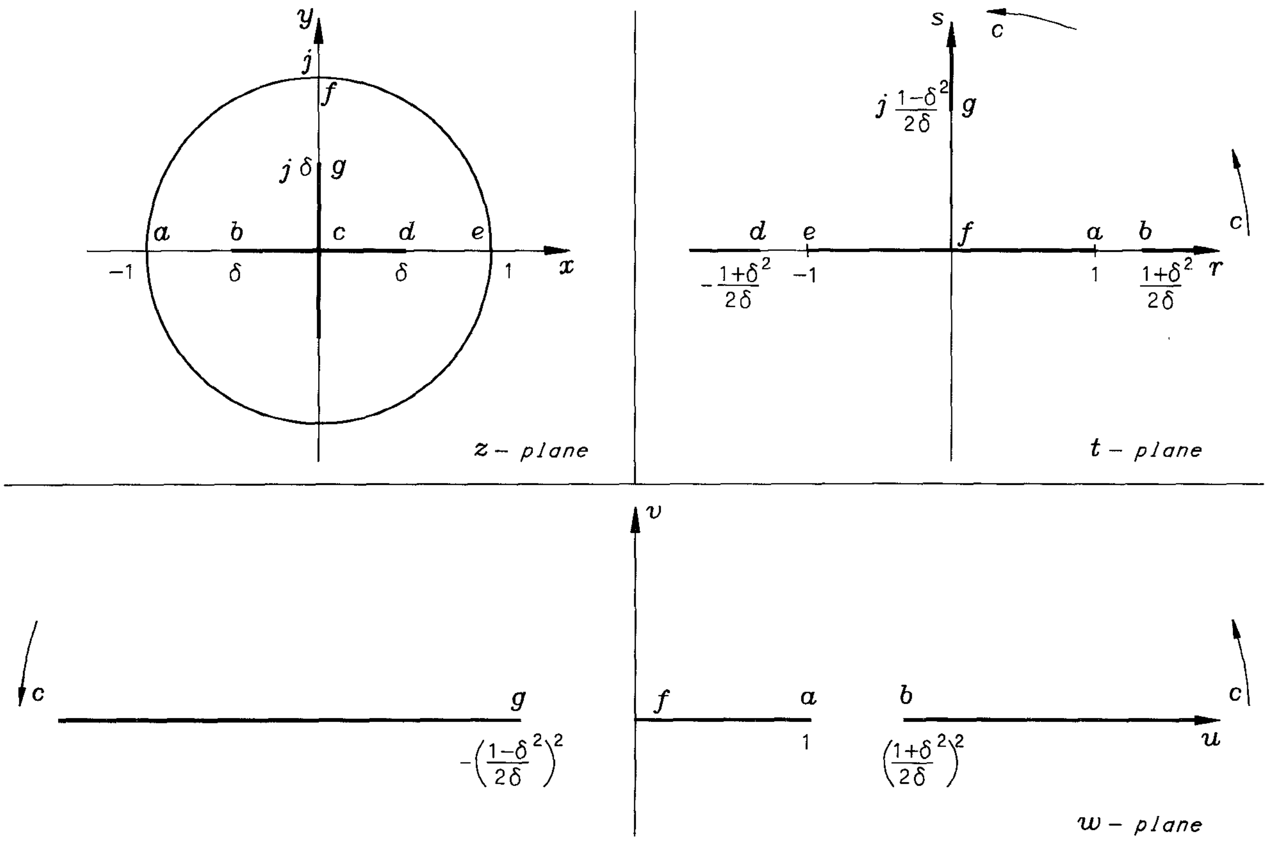

Fig. 1. z, t, and w coordinate planes.

TABLE I
CAPACITANCE OF A CROSS IN A CIRCLE

δ	1	2	3	4	5	6	7	8	9	95
Co(δ)	2 3718	3 2122	4 0523	4 9755	6 0446	7 3372	8 9817	11 2514	15.0037	18 5421
Co	2 3718	3.2122	4 0523	4 9755	6 0446	7 3372	8 9817	11 2514	15.0099	18 6617
Co(ρ)				4 9584	6 0401	7 3364	8 9816	11 2514	15 0099	18 6617
ρ				6	5	4	3	2	1	05

and

$$k'^2 = \rho^4\left(1+2\rho+\frac{3}{2}\rho^2-\frac{1}{3}\rho^3-\frac{39}{16}\rho^4 -\frac{11}{4}\rho^5-\frac{5}{4}\rho^6+\frac{7}{8}\rho^7+\cdots\right). \quad (11)$$

Substitution of (10) in (8) then gives

$$\pi\frac{K}{K'} = 4\ln 2 - 4\ln\rho - 2\rho + \frac{1}{2}\rho^2 + \frac{5}{6}\rho^3 + \frac{1}{16}\rho^4 - \frac{31}{40}\rho^5 - \frac{71}{96}\rho^6 + \frac{41}{224}\rho^7 + \cdots\Big). \quad (12)$$

Returning then to (5), we have

$$C_0(\delta) = 4\pi\Big/\left[\ln 2 - 2\ln\delta - \frac{1}{8}\left(\delta^8 + \frac{13}{32}\delta^{16} + \frac{23}{96}\delta^{24} + \frac{2701}{16384}\delta^{32} + \cdots\right)\right] \quad (13)$$

and

$$C_0(\rho) = \frac{4}{\pi}\left(4\ln 2 - 4\ln\rho - 2\rho + \frac{1}{2}\rho^2 + \frac{5}{6}\rho^3 + \frac{1}{16}\rho^4 - \frac{31}{40}\rho^5 - \frac{71}{96}\rho^6 + \frac{41}{224}\rho^7 + \cdots\right). \quad (14)$$

The upper row of data in Table I gives the values of C_0 obtained from (13) while the lower row of the table gives values obtained from (14). The expansion (13) gives values of C_0 with an error less than 0.05 percent for values of δ less than 0.9 while (14) gives values of C_0 with an error less than 0.02 percent for values of ρ greater than 0.6.

III. AN APPLICATION

The coefficients in the expansion of $\pi K'/K$ in powers of k^2 given in [3, p. 1219], together with the coefficient of k^{24}, are required to determine the first three powers of δ^8 in (9). In order to obtain an additional confirmation of the identity of the coefficients in the expansions (8) and (9), the coefficients of k^{26}, k^{28}, k^{30}, and k^{32}, as given in the Appendix, were found. These permitted the determination of the coefficent of δ^{32} in (9).

It is clear that the expansion of $\pi K'/K$ in powers of k is the same as its expansion in powers of δ^4 except for a factor of 4. This unexpected result can be shown as follows.

The expression for k^2, (6), can be rewritten

$$k = \frac{2\sqrt{k_1}}{1+k_1} \quad (15)$$

if $k_1 = 2\delta^2/(1+\delta^4)$. In turn, $k_1 = 2\sqrt{k_2}/(1+k_2)$ if $k_2 = \delta^4$. Then from [2, p. 73],

$$\frac{K'(k)}{K(k)} = \frac{1}{2}\frac{K'(k_1)}{K(k_1)} = \frac{1}{4}\frac{K'(k_2)}{K(k_2)} = \frac{1}{4}\frac{K'(\delta^4)}{K(\delta^4)} \quad (16)$$

and (9) is established.

Thus the relationship between k and δ, provided by (6), establishes a recurrence relationship between the coefficients of

k^2 in the expansion of $\pi K'/K$. The coefficients of degree, $8(n-3)$, $8(n-2)$, $8(n-1)$, and $8n$, are determined by the coefficients of degree $\leqslant 2n$. That this relationship provides of convincing check on the accuracy of the expansion of $\pi K'/K$, in powers of k^2, is shown in the Appendix.

IV. Extensions

A similar argument applies to the case considered by Oberhettinger and Magnus [1, p. 61] when the strip inner conductor is centrally located. Now we determine the capacitance of the line segment *ea* with respect to the infinite line segment *bd* in the *t* plane of Fig. 1. Using (4), it is found that k is given by

$$k^2 = \frac{8\delta(1+\delta^2)}{(1+\delta)^4}. \tag{17}$$

This differs from (6) only in that δ^2 has been replaced with δ. It follows then from (16) that, for this case,

$$\frac{K'(k)}{K(k)} = \frac{1}{4}\frac{K'(\delta^2)}{K(\delta^2)}. \tag{18}$$

Finally

$$C_0 = 2\frac{K(k)}{K'(k)} = 8\frac{K(\delta^2)}{K'(\delta^2)}. \tag{19}$$

The reader will find it interesting to compare this solution with that given by Nehari [4, p. 293].

The capacitance of a line segment, one of whose ends falls on the origin, is given by the same expansion. We are now concerned with the capacitance in the upper half t plane between the line segment *ea* and the infinite line segment *cd*. For this case, using (4),

$$k^2 = \frac{\delta}{(1+\delta)^2}. \tag{20}$$

It follows that

$$\frac{K'(k)}{K(k)} = \frac{1}{2}\frac{K'(\delta)}{K(\delta)} \tag{21}$$

and

$$C_0 = 4\frac{K(\delta)}{K'(\delta)}. \tag{22}$$

Appendix

Determining (9) from the power series for k^2 required additional terms in the expansion for $K(k)/K'(k)$ given in [3, p. 1219]. These terms are

$$\frac{K'(k)}{K(k)} = \frac{1}{2}\cdots + \frac{1057889591339}{26388279066624}k^{24} + \frac{2069879045935}{57174604644352}k^{26} + \frac{32456762953369}{985162418487296}k^{28} + \frac{63713529525287}{2111062325329920}k^{30} + \frac{512963507737401997}{18446744073709551616}k^{32} + \cdots. \tag{A1}$$

The coefficients in (9) were obtained by using all the coefficients in (A1), including of course those of [3, p. 1219], in (6), (7), and (8). The fact that the coefficients in (9) are known to be correct from (16) is a very convincing indication of the accuracy of (A1). Moreover, it has been found that changing the value of the coefficient of k^{32} in (A1) by only one digit in the last place will alter the coefficient of δ^{32} in (9).

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Equivalent Circuits for Dielectric Posts in a Rectangular Waveguide

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Abstract —**Scattering by dielectric posts in a rectangular waveguide is investigated by a combination of the finite and boundary element methods (CFBEM), and the equivalent circuits are derived. Some of the lossless dielectric post resonances in a rectangular waveguide can be physically realized by a lattice circuit, and the interaction between two posts can be evaluated by this circuit.**

I. Introduction

Recently, studies on scattering in waveguides containing dielectric posts have been reported [1]–[10]. Dielectric resonators are simple and small in size and ceramic dielectrics are readily available, so that ceramic dielectrics with high relative permittivity and temperature stability have been used to design microwave filters. Sahalos and Vafiadis [4] have considered the design of bandpass or band-stop filters, and Gesche and Löchel [7] a tunable band-stop filter. Hsu and Auda [9] have physically realized lossy dielectric post resonance by a lumped network. We expect that an equivalent circuit which can represent the dielectric resonance in a rectangular waveguide would be useful for the design of microwave filters.

We show that some of the lossless dielectric post resonances in a rectangular waveguide can be physically realized by the lumped networks. The interaction between two posts in a rectangular waveguide can be evaluated by these lumped networks.

II. Methods

The problems are analyzed by a combination of the finite and the boundary element method (CFBEM) [10].

The equivalent circuit for a post in a rectangular waveguide is shown in terms of two-port networks. The T network is commonly used. However, it is found that the shunt arm reactance of the T network decreases with increasing frequency [5], [9]. In the case where the post structures are symmetrical about some plane perpendicular to the axis of the transmission line, we can use the lattice network [9], [13]. It has been shown in [13] that any two-port network is physically realizable in the lattice form; i.e., it is unnecessary to use any negative inductances or capacitances to construct the lattice.

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